Fundamental Theorems for Free

Logical Relations & Parametricity for Substructural Type Theory and Beyond

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In this talk, I will outline some of my own recent work on developing logical relations to prove parametricity theorems for substructural type systems, using this as a jumping-off point to discuss a more general categorical *recipe* for logical relations, that can be used to derive these and other examples.

Simply-Typed λ-Calculus

Simply-Typed λ-Calculus (STLC) is the internal language of Cartesian Closed Categories (CCCs), i.e. there is an equivalence of categories:

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- Syn constructs a "syntactic category" for each theory in STLC.
- $\mathcal{L}$ defines a theory in STLC—the *internal language* of $\mathcal{C}$—for each Cartesian Closed Category $\mathcal{C}$.

Natural Number Objects & System T

A **natural number object** in a CCC $\mathcal{C}$, if one exists, is the universal object $\mathbb{N} \in \mathcal{C}$ equipped with morphisms

$$\mathbf{z} : 1 \to \mathbb{N} \quad \text{and} \quad \mathbf{s} : \mathbb{N} \to \mathbb{N}$$

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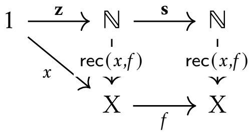

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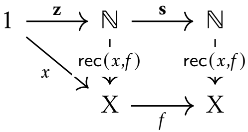

System T (Gödel, 1958) adds to STLC a type of natural numbers $\mathbb{N}$, which makes $\mathsf{Syn}(\textbf{SysT})$ the initial object in the category $\mathbf{CCC}_{\mathbb{N}}$ of CCCs equipped with natural number objects and functors between them that (strictly) preserve finite products, exponentials, and natural number objects.

Canonicity for System T

For each (metatheoretic) natural number m, let $\overline{m}$ be defined by

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Proof: by a logical relations argument.

Logical Relations for System T

For each type τ in System T, let $[\![\tau]\!]$ be the set of closed terms of type τ, quotiented up to judgmental equality. Equivalently, this is $\mathrm{Hom}_{\mathsf{Syn}(\mathbf{SysT})}(1, \tau)$.

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Fundamental Theorem (FTLR): for every open term $\Gamma \vdash a : A$ in System T

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Canonicity then follows as a corollary of the fundamental theorem: for any term $n : \mathbb{N}$, we have $\mathbb{P}_{\mathbb{N}}(n)$, i.e. $n \equiv \overline{m}$ for some m.

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We call a term $f : A[X] \to B[X]$ in $\lambda[X]$ a (unary) *polymorphic function*, since for any type C in any λ-theory $\mathbb{T}$, we can obtain a function

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Reynolds (1983): Polymorphic functions should preserve all predicates/relations definable on types.

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Proof: induction on the derivation of $\Gamma[X] \vdash a : A[X]$

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- It follows that

$$\alpha[C/X](c) \equiv c$$

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Consequences of Parametricity, part 2

Further Example: every polymorphic function $X \to X \to X \times X$ is extensionally equivalent to one of the following four functions:

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- As before, we can unfold the parametricity theorem for α to the following: for any type C in any theory $\mathbb{T}$ and any predicate $P \subseteq [\![C]\!]$

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Consequences of Parametricity, part 2

Further Example: every polymorphic function $X \to X \to X \times X$ is extensionally equivalent to one of the following four functions:

$$\lambda x.\lambda y.(x, y) \quad \lambda x.\lambda y.(y, x) \quad \lambda x.\lambda y.(x, x) \quad \lambda x.\lambda y.(y, y)$$

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- Hence for any $c_0, c_1 : C$, we can take $P = \{c_0, c_1\}$, by which it follows that

$$\pi_1(\alpha[C/X](c_0)(c_1)) \in \{c_0, c_1\} \quad \text{and} \quad \pi_2(\alpha[C/X](c_0)(c_1)) \in \{c_0, c_1\}$$

Monoidal Categories

A **monoidal category** is a (pseudo)monoid object in **Cat**, i.e. a category $\mathcal{M}$ equipped with

$$\mathrm{I} \in \mathcal{M} \quad \text{and} \quad \otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$$

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- A monoid can be regarded as monoidal category with trivial morphisms, and likewise for commutative monoids and SMCs.
- A Cartesian Closed Category canonically carries the structure of a closed SMC with the monoidal structure given by finite products.

Substructural λ-Calculus

We may obtain *substructural* variants of STLC by restricting the usage of variables in forming terms. Specifically:

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We may obtain *substructural* variants of STLC by restricting the usage of variables in forming terms. Specifically:

- **Linear λ-calculus** requires each variable to be used exactly once.
- **Ordered λ-calculus** additionally requires variables to be used in the order in which they occur in the typing context.

Ordered λ-calculus may then be characterized as the internal language of *biclosed monoidal categories*, and linear λ-calculus as the internal language of *closed symmetric monoidal categories*, in that there are equivalences

$$\mathbf{STLC}^{\mathbf{Ord}} \underset{\mathcal{L}}{\overset{\mathsf{Syn}}{\underset{\simeq}{\rightleftarrows}}} \mathbf{MonCat}^{\mathbf{Biclosed}}$$

and

$$\mathbf{STLC}^{\mathbf{Lin}} \underset{\mathcal{L}}{\overset{\mathsf{Syn}}{\underset{\simeq}{\rightleftarrows}}} \mathbf{SMC}^{\mathbf{Closed}}$$

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We could try to repeat the proof of parametricity for ordinary STLC by considering (e.g.) the theory $\lambda^{Ord}[X]$ of a type X in ordered STLC and defining logical relations for this theory as before. However, this gives us no additional information about terms in ordered λ-calculus beyond what could already be deduced from the parametricity theorem for ordinary λ-calculus.

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For instance, we can apply the same argument as before to show that any polymorphic function $\alpha : \mathrm{X} \to \mathrm{X} \to \mathrm{X} \otimes \mathrm{X}$ in ordered λ-calculus must be equivalent to one of the following four functions

$$\lambda x.\lambda y.(x, y) \quad \lambda x.\lambda y.(y, x) \quad \lambda x.\lambda y.(x, x) \quad \lambda x.\lambda y.(y, y)$$

but we cannot deduce the stronger (but correct) result that in fact α must be equivalent to only the first of these.

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m \Vdash_{A[X] \twoheadrightarrow B[X]} f &\iff \forall n \in M,\ a : A[C], \\
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FTLR for Ordered STLC: For any type $\mathbb{C}$ in a theory $\mathbb{T}$ in Ordered STLC, if $\Gamma[X] \vdash a : A[X]$ is an open term in $\lambda^{Ord}[X]$, and $(M, \epsilon, \cdot)$ is *any* monoid, then

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Proof (both): induction on the derivation of $\Gamma[X] \vdash a : A[X]$

Applications of Substructural Parametricity

Theorem: every polymorphic function $F : \mathrm{X} \multimap \mathrm{X} \multimap \mathrm{X} \otimes \mathrm{X}$ in ordered STLC must be extensionally equivalent to

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- Substituting α for m and β for n, this implies that we have $m', n' \in \{\alpha, \beta\}$ such that $\alpha\beta = m'n'$, which implies that $m' = \alpha$ and $n' = \beta$, and therefore $c_0' = c_0$ and $c_1' = c_1$, i.e.

$$F[C/X] c_0\ c_1 \equiv \langle c_0, c_1\rangle$$

How did I do that?

From Logical to Categorical Relations

Question: Can we derive parametricity from properties of $\mathsf{Syn}_{\lambda^{Ord}[X]}$, rather than going via laborious inductions on syntax?

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Idea: given

- a monoid $(M, \epsilon, \cdot)$,
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construct a pointed biclosed monoidal “Category of Relations” $\mathsf{Rel}_{M,C,R}$, and derive parametricity from the existence of a (strict) biclosed monoidal functor

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But how to construct $\mathsf{Rel}_{M,C,R}$?

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- $\pi(\mathrm{A}, p_{\mathrm{A}}) = \mathrm{A}$ and $\pi(f) = f$.

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A **lax monoidal copresheaf (LMC)** on a monoidal category $\mathcal{M}$ is a functor $\mathcal{M} \to \mathbf{Set}$ equipped with coherent natural transformations:

$$\epsilon : 1 \to \Gamma(I) \quad \text{and} \quad (\cdot) : \Gamma(A) \times \Gamma(B) \to \Gamma(A \otimes B)$$

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- If $\Gamma : \mathcal{N} \to \mathbf{Set}$ is an LMC on $\mathcal{N}$, and $F : \mathcal{M} \to \mathcal{N}$ is a monoidal functor, then the precomposition of Γ with F carries the structure of an MDO on $\mathcal{M}$:

  $$\mathcal{M} \xrightarrow{F} \mathcal{N} \xrightarrow{\Gamma} \mathbf{Set}$$

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Call this the **category of relations** Rel_Γ of Γ.

Day Convolution

Theorem (Day Convolution): given an LMC $\Gamma : \mathcal{M} \to \mathbf{Set}$ on a biclosed monoidal category $\mathcal{M}$, Rel_Γ carries the structure of a (strict) biclosed monoidal category over $\mathcal{M}$.

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- For A, B and $\mathrm{P_A}, \mathrm{P_B}$ as above, the left closure $(\mathrm{A}, \mathrm{P_A}) \multimap (\mathrm{B}, \mathrm{P_B})$ is

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- And similarly for the right closure.

Putting it all together

Given a theory $\mathbb{T}$ together with a type C in $\mathbb{T}$, and a monoid M together with with a relation $R \subseteq M \times [\![C]\!]$, define Γ to be the following composite

$$\Gamma \quad := \quad \mathsf{Syn}_{\lambda^{Ord}[X]} \xrightarrow{[C/X]} \mathsf{Syn}_{\mathbb{T}} \xrightarrow{Hom_{\mathsf{Syn}_{\mathbb{T}}}(I,-)\times M} \mathbf{Set}$$

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Applying Day Convolution yields a biclosed monoidal structure on

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Moreover, we have $(X, R) \in \mathsf{Rel}_{\Gamma}$. Hence there is a pointed biclosed monoidal functor

$$\Vdash_{(-)} : \mathsf{Syn}_{\lambda^{Ord}[X]} \longrightarrow \mathsf{Rel}_{\Gamma}$$

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Hence the composite $\pi \circ \Vdash_{(-)} : \mathsf{Syn}_{\lambda^{Ord}[X]} \to \mathsf{Syn}_{\lambda^{Ord}[X]}$ is also a pointed biclosed monoidal functor. But since $\mathsf{Syn}_{\lambda^{Ord}[X]}$ is the initial pointed biclosed monoidal category it follows that this must be the identity on $\mathsf{Syn}_{\lambda^{Ord}[X]}$, i.e.

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Unpacking this, by functoriality of $\Vdash_{(-)}$, for all $f : A[X] \longrightarrow B[X] \in \mathsf{Syn}_{\lambda^{Ord}}[X]$ we have that

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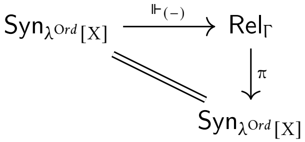

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which precisely FLTR.

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3. Derive FTLR from the existence of a section of this projection, which follows from initiality of $\mathsf{Syn}_{\mathbb{T}}$ as below:

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2. How do the forms of *external* parametricity considered in this talk relate to type theories with *internal* parametricity and their categorical semantics? (Next time!)

Thank you!