Fundamental Theorems for Free

Logical Relations & Parametricity for Substructural Type Systems and Beyond

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What is a Logical Relation?

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In this talk, I will first sketch some of my own recent work on developing logical relations to prove parametricity theorems for substructural type systems, using this as a jumping-off point to discuss a more general *recipe* for logical relations, based on category theory, that can be used to derive these and other examples.

Recap: Simply-Typed λ-Calculus

Our starting point for most of the type systems considered in this talk will be the Simply-Typed λ-Calculus with both function and pair types:

$$\frac{}{\mathbf{1}\ \mathsf{Type}} \qquad \frac{A\ \mathsf{Type} \qquad B\ \mathsf{Type}}{A \times B\ \mathsf{Type}} \qquad \frac{A\ \mathsf{Type} \qquad B\ \mathsf{Type}}{A \longrightarrow B\ \mathsf{Type}}$$

$$\frac{}{\Gamma, x : A, \Gamma' \vdash x : A} \qquad \frac{}{\Gamma \vdash () : \mathbf{1}}$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash (a, b) : A \times B} \qquad \frac{\Gamma \vdash p : A \times B}{\Gamma \vdash \pi_1(p) : A} \qquad \frac{\Gamma \vdash p : A \times B}{\Gamma \vdash \pi_2(p) : B}$$

$$\frac{\Gamma, x : A \vdash f : B}{\Gamma \vdash \lambda x.f : A \longrightarrow B} \qquad \frac{\Gamma \vdash f : A \longrightarrow B \qquad \Gamma \vdash a : A}{\Gamma \vdash f(a) : B}$$

$$\pi_1(a, b) \equiv_A a \qquad \pi_2(a, b) \equiv_B b \qquad p \equiv_{A \times B} (\pi_1(p), \pi_2(p))$$

$$(\lambda x.f)(a) \equiv_B f[a/x] \qquad f \equiv_{A \to B} \lambda x.f(x) \qquad u \equiv_{\mathbf{1}} ()$$

Example 1: System T

System T extends simply-typed λ-calculus with a type of natural numbers:

$$\frac{}{\mathbb{N}\ \mathsf{Type}} \qquad \frac{}{\Gamma \vdash 0 : \mathbb{N}} \qquad \frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash \mathbf{s}(n) : \mathbb{N}}$$

$$\frac{\Gamma \vdash n : \mathbb{N} \qquad \Gamma \vdash a_0 : \mathrm{A} \qquad \Gamma, x : \mathbb{N}, y : \mathrm{A} \vdash a_1 : \mathrm{A}}{\Gamma \vdash \mathsf{rec}\ n\ \{0 \mapsto a_0 \mid \mathbf{s}(x), y \mapsto a_1\} : \mathrm{A}}$$

$$\mathsf{rec}\ 0\ \{0 \mapsto a_0 \mid \dots\} \equiv_{\mathrm{A}} a_0$$

$$\mathsf{rec}\ \mathbf{s}(n)\ \{\cdots \mid \mathbf{s}(x), y \mapsto a_1\} \equiv_{\mathrm{A}} a_1[n/x, \mathsf{rec}\ n\ \{\dots\}/y]$$

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For any natural number m, let $\overline{m} : \mathbb{N}$ be defined inductively as follows:

$$\begin{aligned} \overline{0} &= 0 \\ \overline{m+1} &= \mathbf{s}(\overline{m}) \end{aligned}$$

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Then for each τ Type, define a predicate $\mathbb{P}_\tau \subseteq [\![\tau]\!]$, as follows:

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\begin{array}{rcl}
\mathbb{P}_{\mathbb{N}}(n) & \iff & \exists m \text{ s.t. } n \equiv_{\mathbb{N}} \overline{m} \\
\mathbb{P}_{A \times B}(p) & \iff & \mathbb{P}_A(\pi_1(p)) \text{ and } \mathbb{P}_B(\pi_2(p)) \\
\mathbb{P}_{A \to B}(f) & \iff & \forall a \in [\![A]\!].\mathbb{P}_A(a) \implies \mathbb{P}_B(f(a))
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Fundamental Theorem (FTLR): for every $\Gamma \vdash a : A$ and $\gamma : \Gamma$

$$\text{if } \mathbb{P}_\Gamma(\gamma) \text{ then } \mathbb{P}_A(a[\gamma/\Gamma])$$

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Proof: induction on derivations.

Canonicity then follows as a corollary of the fundamental theorem.

Example 2: Polymorphism & System F

System F extends simply-typed λ-calculus with parametric polymorphism:

$$\frac{}{\Delta, X, \Delta' \mid X\ \mathsf{Type}} \qquad \frac{\Delta, X \mid A\ \mathsf{Type}}{\Delta \mid \forall X.A\ \mathsf{Type}}$$

$$\frac{\Delta, X \mid \Gamma \vdash F : A}{\Delta \mid \Gamma \vdash \Lambda X.F : \forall X.A} \qquad \frac{\Delta \mid \Gamma \vdash F : \forall X : A \qquad \Delta \mid B\ \mathsf{Type}}{\Delta \mid \Gamma \vdash F[B] : A[B/X]}$$

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Intuitively, Polymorphic functions in System F can't inspect the types over which they are defined and so must behave uniformly for all types at which they are instantiated. But how to make this idea precise?

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This idea is then made precise using a logical relations construction.

(Unary) Parametricity for System F

For each System F type $X_1, \ldots, X_n \mid A\ \mathsf{Type}$, given types $B_1, \ldots, B_n$ and predicates $P_i \subseteq [\![B_i]\!]$ for $i = 1, \ldots, n$, define a predicate

$$\mathbb{P}_A^{X_i \mapsto P_i} \subseteq [\![A[B_1/X_1, \ldots, B_n/X_n]]\!]$$

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as follows:

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\mathbb{P}_{X_j}^{X_i \mapsto P_i}(x) &\iff P_j(x) \\
\mathbb{P}_{\mathbf{1}}^{X_i \mapsto P_i}(u) &\iff u \equiv_{\mathbf{1}} () \\
\mathbb{P}_{A \times B}^{X_i \mapsto P_i}(p) &\iff \mathbb{P}_A^{X_i \mapsto P_i}(\pi_1(p)) \text{ and } \mathbb{P}_B^{X_i \mapsto P_i}(\pi_2(p)) \\
\mathbb{P}_{A \to B}^{X_i \mapsto P_i}(f) &\iff \forall a : A.\ \mathbb{P}_A^{X_i \mapsto P_i}(a) \implies \mathbb{P}_B^{X_i \mapsto P_i}(f(a)) \\
\mathbb{P}_{\forall X.A}^{X_i \mapsto P_i}(F) &\iff \forall B \text{ Type},\ P \subseteq [\![B]\!].\ \mathbb{P}_A^{X_i \mapsto P_i,\ X \mapsto P}(F[B])
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FTLR: for all $X_1, \ldots, X_n \mid \Gamma \vdash a : A$, given closed types $B_1, \ldots, B_n$ with predicates $P_1, \ldots, P_n$ as above, and $\gamma : \Gamma$

$$\text{if } \mathbb{P}_\Gamma^{X_i \mapsto P_i}(\gamma) \text{ then } \mathbb{P}_A^{X_i \mapsto P_i}(a[\gamma/\Gamma])$$

Consequences of Parametricity, part 1

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- Therefore $\mathbb{P}^{X \mapsto P}_{X}(\alpha[A](a))$ – i.e. $P(\alpha[A](a))$ – for all $a : A$ such that $P(a)$.

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- Therefore $\mathbb{P}^{X \mapsto P}_{X}(\alpha[A](a))$ – i.e. $P(\alpha[A](a))$ – for all $a : A$ such that $P(a)$.
- Hence for any closed type A and $a : A$, we can define $P \subseteq [\![A]\!]$ by $P = \{a\}$. By construction $b \in P \iff b \equiv_A a$, and so by the above it follows that $\alpha[A](a) \equiv_A a$.

Consequences of Parametricity, part 2

Further Example: every closed term $\alpha : \forall X.X \to X \to X \times X$ is extensionally equivalent to one of the following four functions:

$$\Lambda X.\lambda x.\lambda y.(x, y) \quad \Lambda X.\lambda x.\lambda y.(y, x) \quad \Lambda X.\lambda x.\lambda y.(x, x) \quad \Lambda X.\lambda x.\lambda y.(y, y)$$

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Proof:

- As before, we can unfold the parametricity theorem for $\alpha : \forall X.X \to X \to X \times X$ to the following:

$$\begin{array}{l} \forall A\ \mathsf{Type},\ P \subseteq [\![A]\!],\ a_0, a_1 : A.\ P(a_0) \text{ and } P(a_1) \\ \quad \Longrightarrow\ P(\pi_1(\alpha[A](a_0)(a_1))) \text{ and } P(\pi_2(\alpha[A](a_0)(a_1))) \end{array}$$

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- Hence for any A Type with $a_0, a_1 : A$, we can take $P = \{a_0, a_1\}$, by which it follows that

$$\pi_1(\alpha[A](a_0)(a_1)) \in \{a_0, a_1\} \quad \text{and} \quad \pi_2(\alpha[A](a_0)(a_1)) \in \{a_0, a_1\}$$

Ordered STLC

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We then have the following modified types and rules, which now must preserve the relative order and multiplicities of variables in contexts:

$$\frac{}{\mathbf{1}\ \mathsf{Type}} \qquad \frac{\mathrm{A}\ \mathsf{Type} \qquad \mathrm{B}\ \mathsf{Type}}{\mathrm{A} \otimes \mathrm{B}\ \mathsf{Type}} \qquad \frac{\mathrm{A}\ \mathsf{Type} \qquad \mathrm{B}\ \mathsf{Type}}{\mathrm{A}/\mathrm{B}\ \mathsf{Type}} \qquad \frac{\mathrm{A}\ \mathsf{Type} \qquad \mathrm{B}\ \mathsf{Type}}{\mathrm{A} \setminus \mathrm{B}\ \mathsf{Type}}$$

$$\frac{}{\vdash \langle\rangle : \mathbf{1}} \qquad \frac{\Delta \vdash u : \mathbf{1} \qquad \Gamma, \Theta \vdash c : \mathrm{C}}{\Gamma, \Delta, \Theta \vdash \mathsf{let}\ \langle\rangle = u\ \mathsf{in}\ c : \mathrm{C}}$$

$$\frac{\Gamma \vdash a : \mathrm{A} \qquad \Delta \vdash b : \mathrm{B}}{\Gamma, \Delta \vdash \langle a, b\rangle : \mathrm{A} \otimes \mathrm{B}} \qquad \frac{\Delta \vdash p : \mathrm{A} \otimes \mathrm{B} \qquad \Gamma, x : \mathrm{A}, y : \mathrm{B}, \Theta \vdash c : \mathrm{C}}{\Gamma, \Delta, \Theta \vdash \mathsf{let}\ \langle x, y\rangle = p\ \mathsf{in}\ c : \mathrm{C}}$$

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Substructural Logic & Resources

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We can represent such resources as a *monoid*, i.e. a set M equipped with an associative binary operation $\otimes : \mathrm{M} \times \mathrm{M} \longrightarrow \mathrm{M}$ and a unit element $\epsilon \in \mathrm{M}$, i.e. subject to the following equations

$$\epsilon \otimes m = m = m \otimes \epsilon \qquad (k \otimes m) \otimes n = k \otimes (m \otimes n) \qquad \forall k, m, n \in \mathrm{M}$$

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A common interpretation of substructural logic & type theory is as a logic of *resources*, where the rules of the logic/type theory reflect the ways in which resources may be created, transformed, and consumed.

We can represent such resources as a *monoid*, i.e. a set M equipped with an associative binary operation $\otimes : \mathrm{M} \times \mathrm{M} \longrightarrow \mathrm{M}$ and a unit element $\epsilon \in \mathrm{M}$, i.e. subject to the following equations

$$\epsilon \otimes m = m = m \otimes \epsilon \qquad (k \otimes m) \otimes n = k \otimes (m \otimes n) \qquad \forall k, m, n \in \mathrm{M}$$

Intuitively, $\otimes$ allows us to *accumulate* resources, and ϵ represents *no resource*.

Substructural Logical Relations

Fix a monoid M. For each type A Type in substructural STLC, rather than a mere predicate $P \subseteq [\![A]\!]$, we define a relation $\Vdash_A \subseteq M \times [\![A]\!]$ – where $m \Vdash_A a$ roughly means that a satisfies the logical predicate in the presence of m resources – inductively as follows:

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$$
\begin{aligned}
m \Vdash_{\mathbf{1}} u \quad &\Longleftrightarrow \quad m = \epsilon \text{ and } u \equiv_{\mathbf{1}} \langle\rangle \\
m \Vdash_{A \otimes B} p \quad &\Longleftrightarrow \quad \exists n, k \in M,\ a \in [\![A]\!],\ b \in [\![B]\!] \text{ such that} \\
&\qquad m = n \otimes k,\ p \equiv_{A \otimes B} \langle a, b \rangle,\ n \Vdash_A a,\ \text{and } k \Vdash_B b \\
m \Vdash_{A/B} f \quad &\Longleftrightarrow \quad \forall n \in M,\ a \in [\![A]\!],\ n \Vdash_A a \implies n \otimes m \Vdash_B f(a) \\
m \Vdash_{B \backslash A} f \quad &\Longleftrightarrow \quad \forall n \in M,\ a \in [\![A]\!],\ n \Vdash_A a \implies m \otimes n \Vdash_B f(a)
\end{aligned}
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Substructural Parametricity

Adding parametric polymorphism to substructural STLC works exactly the same as for System F.

$$\frac{}{\Delta, X, \Delta' \mid X \; \mathsf{Type}} \qquad \frac{\Delta, X \mid A \; \mathsf{Type}}{\Delta \mid \forall X.A \; \mathsf{Type}}$$

$$\frac{\Delta, X \mid \Gamma \vdash F : A}{\Delta \mid \Gamma \vdash \Lambda X.F : \forall X.A} \qquad \frac{\Delta \mid \Gamma \vdash F : \forall X : A \qquad \Delta \mid B \; \mathsf{Type}}{\Delta \mid \Gamma \vdash F[B] : A[B/X]}$$

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But now we define parametricity for these types with respect to all relations $R \subseteq M \times [\![A]\!]$, rather than just predicates on closed terms.

$$m \Vdash_{X_j}^{X_i \mapsto R_i} x \iff m\ R_j\ x$$

$$m \Vdash_{\forall X.A}^{X_i \mapsto R_i} F \iff \forall B\ \mathsf{Type},\ R \subseteq M \times [\![B]\!],\ m \Vdash_{A}^{X_i \mapsto R_i,\ X \mapsto R} F[B]$$

The Fundamental Theorem

FTLR: if $X_1, \ldots, X_n \mid \Gamma \vdash a : A$ is derivable in polymorphic ordered STLC, and M is *any* monoid, then:

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Proof: Induction on the derivation of $X_1, \ldots, X_n \mid \Gamma \vdash a : A$.

Applications of Substructural Parametricity

Theorem: every closed term $F : \forall X.X/X/(X \otimes X)$ in ordered STLC must be extensionally equivalent to

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Then for any type A Type with $a_0, a_1 : A$, define $R \subseteq M \times [\![A]\!]$ by $m \mathrel{R} a \iff$ either $m = \alpha$ and $a = a_0$ or $m = \beta$ and $a = a_1$.

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- Substituting α for m and β for n, this implies that we have $m', n' \in \{\alpha, \beta\}$ such that $\alpha\beta = m' \otimes n'$, which implies that $m' = \alpha$ and $n' = \beta$, and therefore $a' = a_0$ and $a'' = a_1$, i.e. $\alpha\ A\ a_0\ a_1 \equiv_A \langle a_0, a_1\rangle$.

How did I do that?

Categories & Languages

Definition: a *category* $\mathcal{C}$ consists of a set of *objects*, and for each pair of objects a set of *arrows* or *morphisms* between them.

$$\mathrm{A}, \mathrm{B} \in \mathcal{C} \qquad f : \mathrm{A} \longrightarrow \mathrm{B} \in \mathcal{C}$$

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$$\mathrm{id}_A : A \longrightarrow A \in \mathcal{C} \qquad \begin{array}{l} f : A \longrightarrow B \in \mathcal{C},\ g : B \longrightarrow C \in \mathcal{C} \\ \vdash g \circ f : A \longrightarrow C \in \mathcal{C} \end{array}$$

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Example: For any type system $\mathbb{T}$, there is a category $\mathsf{Syn}_{\mathbb{T}}$ – the *syntactic category* of $\mathbb{T}$ – whose objects are types in in $\mathbb{T}$ and whose morphisms $\mathrm{A} \longrightarrow \mathrm{B}$ are terms $x : \mathrm{A} \vdash b : \mathrm{B}$ in $\mathbb{T}$, quotiented up to judgmental equality in $\mathbb{T}$.

Universal Properties

One advantage of the language of category theory is that it lets us straightforwardly characterize various mathematical constructions by how they relate to other objects in the same category:

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Initial and terminal objects: an object $\bot$ in a category $\mathcal{C}$ is *initial* if for every other object $A \in \mathcal{C}$ there is a unique morphism $*_A$ from $\bot$ to A. Dually, an object $\top$ is *terminal* if for every object $A \in \mathcal{C}$, there is a unique morphism $!_A$ from A to $\top$.

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Example: The unit type **1** is the terminal object in $\mathsf{Syn}_{\mathrm{STLC}}$.

Products

Given objects $A, B \in \mathcal{C}$, the *product* of A and B is an object $A \times B$ with projection morphisms $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$ such that:

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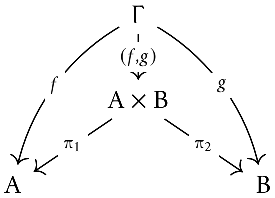

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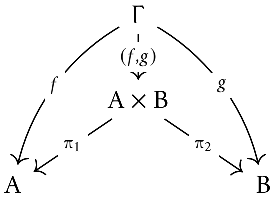

Example: In $\mathsf{Syn}_{\mathrm{STLC}}$, the product of two types A, B is given by the product type $A \times B$.

Exponentials & Natural Numbers Objects

In a category with products $\mathcal{C}$, given two objects $A, B \in \mathcal{C}$, the *exponential* of A and B is an object B^A with a morphism $\alpha : B^A \times A \longrightarrow B$ such that:

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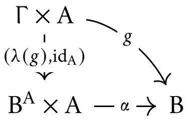

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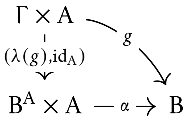

Similarly, if $\mathcal{C}$ additionally has a terminal object $\top$, then a *natural numbers object* (NNO) is an object $\mathbb{N} \in \mathcal{C}$ with $0 : \top \to \mathbb{N}$ and $\mathbf{s} : \mathbb{N} \to \mathbb{N}$ such that:

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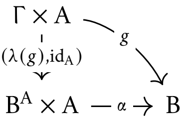

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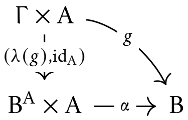

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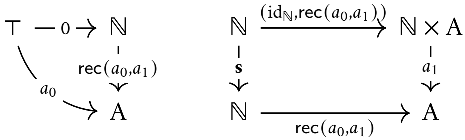
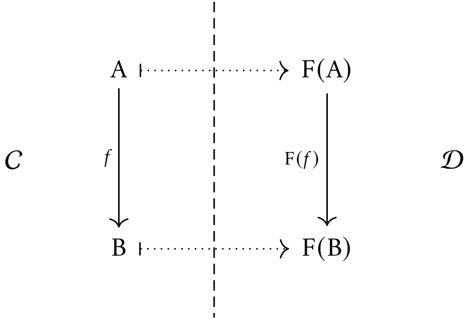


Functors

Given categories $\mathcal{C}, \mathcal{D}$, a *functor* maps objects in $\mathcal{C}$ to objects in $\mathcal{D}$, and morphisms in $\mathcal{C}$ to morphisms in $\mathcal{D}$, as depicted below:



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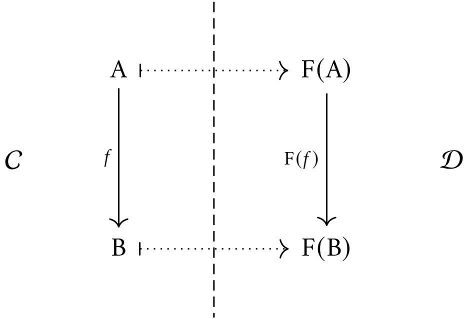

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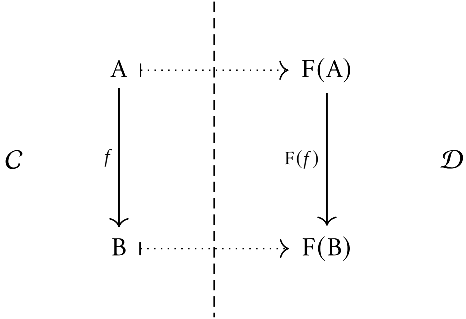

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Note that functors compose associatively and unitally – hence there is a category **Cat** whose objects are categories and whose morphisms are functors.

Initiality of $\mathsf{Syn}_{\mathrm{T}}$

A category $\mathcal{C}$ with a terminal object, products, and exponentials is called *Cartesian Closed*. A functor $\mathrm{F} : \mathcal{C} \to \mathcal{D}$ is called (strictly) Cartesian Closed if it preserves the terminal object, products, and exponentials, i.e.

$$\mathrm{F}(\top) = \top \qquad \mathrm{F}(\mathrm{A} \times \mathrm{B}) = \mathrm{F}(\mathrm{A}) \times \mathrm{F}(\mathrm{B}) \qquad \mathrm{F}(\mathrm{B}^{\mathrm{A}}) = \mathrm{F}(\mathrm{B})^{\mathrm{F}(\mathrm{A})}$$

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Theorem: Syn_T, the syntactic category of System T, is initial in $\mathbf{CCC}^{\mathbb{N}}$.

Initiality of Syn_T

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Similarly, if $\mathcal{C}, \mathcal{D}$ have NNOs, then F (strictly) preserves these if $F(\mathbb{N}) = \mathbb{N}$.

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Proof: induction on derivations OR sledgehammer with general facts about GATs, etc.

Idea: reduce FTLR for System T to the existence of a functor $\mathsf{Syn}_{\mathrm{T}} \to \mathsf{Pred}_{\mathrm{T}} \in \mathbf{CCC}^{\mathbb{N}}$, for a suitably constructed category of “logical predicates” $\mathsf{Pred}_{\mathrm{T}}$.

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- The terminal object of Pred_{T} is the pair $(\mathbf{1}, \{()\})$
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- The natural numbers object of Pred_T is given by $(\mathbb{N}, P_{\mathbb{N}})$ where

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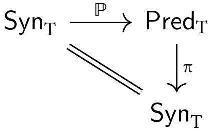

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3. Derive FTLR from the existence of a commuting triangle of functors as below:

$$\begin{array}{ccc} \mathsf{Syn}_{\mathbb{T}} & \longrightarrow & \mathsf{Rel}_{\mathbb{T}} \\ & \diagdown\!\!\diagdown & \big\downarrow \pi \\ & & \mathsf{Syn}_{\mathbb{T}} \end{array}$$

Monoidal Categories & Ordered STLC

A *monoidal category* is a category $\mathcal{M}$ equipped with an object $\epsilon \in \mathcal{M}$ and a functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ with natural isomorphisms

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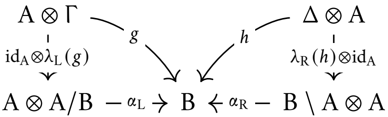

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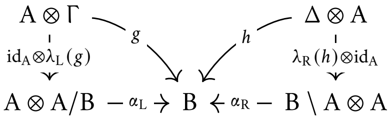

Theorem: The syntactic category of ordered STLC is initial in the category of biclosed monoidal categories and functors that preserve ϵ, $\otimes$, $/$, and $\setminus$.

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- And similarly for the right closure.

Adding Polymorphism

The syntactic category of System F is not really one category, but rather a family of Cartesian Closed categories $\mathsf{Syn}[X_1, \ldots, X_n]$ for each list of type variables $X_1, \ldots, X_n$. These come equipped with functors

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Idea: for each $X_1, \ldots, X_n$, build a Cartesian Closed category $\mathsf{Rel}[X_1, \ldots, X_n]$ following the same general procedure as for STLC/System T/etc., and then show that there are functors $\forall_{X_{n+1}} : \mathsf{Rel}[X_1, \ldots, X_n, X_{n+1}] \to \mathsf{Rel}[X_1, \ldots, X_n]$ with the requisite universal property, using the definition of logical relations for parametrically polymorphic functions.

Adding Polymorphism

The syntactic category of System F is not really one category, but rather a family of Cartesian Closed categories $\mathsf{Syn}[X_1, \ldots, X_n]$ for each list of type variables $X_1, \ldots, X_n$. These come equipped with functors

$$\forall_{X_{n+1}} : \mathsf{Syn}[X_1, \ldots, X_n, X_{n+1}] \to \mathsf{Syn}[X_1, \ldots, X_n]$$

with a certain universal property. Moreover, this family of categories is *initial* with this property.

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This procedure works just as well when we replace Cartesian Closed categories with biclosed monoidal categories, and thus allows us to generalize parametricity from ordinary System F to substructural variants thereof (and even further beyond!)

Conclusion

Category theory provides a general and powerfully simplifying language for logical relations. In this talk, we have seen how this method can be used to handle logical relations and parametricity for ordered polymorphic λ-calculus, but the applications of this method are seemingly without limit.

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Food for thought: what makes a category behave like a “category of relations”?

Thank you!